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Fractional two-branes, toric orbifolds and the quantum McKay correspondence

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Abstract: We systematically study and obtain the large-volume analogues of fractional two-branes on resolutions of the orbifolds $\mathbb{C}^3/\mathbb{Z}_n$. We also study a generalisation of the McKay correspondence proposed in hep-th/0504164 called the quantum McKay correspondence by constructing duals to the fractional two-branes. Details are explicitly worked out for two examples – the crepant resolutions of $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^3/\mathbb{Z}_5$.

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1. Introduction

In an earlier paper [1], hereinafter referred to as I, we had initiated the study of fractional $2p$-branes on non-compact orbifolds. These constructions were a logical extension of the fractional zero-branes on orbifolds which have been well studied. While the actual construction of the boundary states of these fractional $2p$-branes in orbifolds had been done earlier, there had been no detailed study of these branes as part of the general study of B-type branes on the non-compact Calabi-Yau (CY) manifolds in whose Kähler moduli space these orbifolds appeared as special points. In particular there had been no detailed study of the large-volume analogues of these fractional $2p$-branes, apart from some useful preliminary remarks in [2].

While these fractional branes are of interest in their own right, in I we also showed two interesting connections between these fractional branes and other constructions in the study of B-type branes in the Landau-Ginzburg (LG) phase of compact CY manifolds. First we showed that a particular class of fractional two-branes in the LG orbifold, on restriction to the CY hypersurface, were indeed the same as the branes that had been constructed by Ashok et al. [3] using the techniques of boundary fermions and matrix factorisation of the world-sheet superpotential. This related the boundary fermion and matrix factorisation construction to a more straightforward physical construction in terms of simple boundary conditions on the fields of the LG model [4]. Second, we argued that the corresponding conformal field theory boundary states at the LG point in the Kähler moduli space of the CY were in fact the so-called permutation branes [5] of the Gepner construction of the bulk world-sheet conformal field theory. Our argument was further strengthened by the complete computation presented by [6] for these permutation branes(see also [7]).

However the construction of the large-volume analogues of the fractional two-branes in I had relied heavily on some physical arguments. In particular, the coherent sheaves that corresponded to these fractional two-branes appeared as the cohomology of exact sequences that had really had no proper mathematical meaning, as the assignment of fractional 2-brane charges was entirely ad hoc. By independent cohomology and K-theory arguments we had however established that the large-volume analogues of the fractional two-branes were entirely sensible from the point of view of sheaves that were associated to the full non-compact CY manifold. In particular in the full non-compact space the fractional charge had an entirely sensible interpretation as an integer charge in a new basis.[1]

In this paper we re-examine these questions from the point of view of toric geometry. The toric description, as we will see, is particularly useful in the description of sheaves on the full non-compact CY. We will show that the charges of the large-volume analogues of the fractional two-branes, in terms of the Chern characters of these objects, can be determined consistently in the framework of toric geometry. We will also see that this determination shows the heuristic sequences that we wrote down for these objects are also meaningful in a sense that we will explain.

Another aspect of these fractional two-branes that we had studied briefly in I was their relation to the quantum McKay correspondence. One way of stating the classical

[1]The appearance of fractional charges was already noted in [8]

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McKay correspondence is as a relation between the sheaves corresponding to the fractional zero-branes and a dual set of bundles, the so-called tautological bundles, on the full non-compact CY [9 − 11]. For fractional $2p$ branes we conjectured a new quantum McKay correspondence where a new set of sheaves dual to the fractional $2p$-branes play the role of the tautological bundles for the fractional zero-branes. In this paper, we use the toric description to determine these dual objects carefully for fractional two-branes on $\mathbb{C}^3/\mathbb{Z}_n$ orbifolds.

The organisation of the paper is as follows: In section 2, we discuss aspects of local Chern characters that one needs to work with in the context of non-compact manifolds. In sections 3 and 4, we systematically work out the Chern characters for coherent sheaves associated with fractional zero- and two-branes for the resolutions of the orbifolds, $\mathbb{C}^3/\mathbb{Z}_3$ and $\mathbb{C}^3/\mathbb{Z}_5$. This is done by using the open-string Witten index computed in conformal field theory as input along with an ansatz for the form of the Chern character of the coherent sheaves. We fix an ambiguity that arises due to linear equivalences and then present concrete objects that reproduce the computed Chern character. In section 5, we discuss the quantum McKay correspondence and work out the Chern character for sheaves that are dual to the fractional two-branes and finally propose candidate objects for the dual sheaves that are consistent with the computed Chern character. We present our conclusions in section 6. Some of the details of the computations have been presented in four appendices.

2. Background

The gauged linear sigma model has provided a concrete model which enables one to interpolate between orbifolds and their resolution. The complexified Fayet-Iliopoulos parameters are the blow-up moduli and at "large volume" give the sizes of various cycles. In the presence of a boundary preserving B-type supersymmetry, at large volume, the D-branes are best described as coherent sheaves while at the orbifold point one can construct boundary states. Relating these two different descriptions has lead to a surprising connection to the McKay correspondence for fractional zero-branes [12 − 14].

The main goal of this paper is to obtain a systematic understanding of coherent sheaves that are obtained by analytic continuation of fractional two-branes from the orbifold point to large-volume. The construction of the fractional two-branes as boundary states in the orbifold is standard. The open-string Witten index is independent of this analytic continuation and provides an important input in identifying the relevant coherent sheaves. However, this data is not enough to reconstruct even the Chern character (equivalently, the RR charges). An additional complication is that the fractional two-branes are non-compact objects and this has to be dealt with as well. As was shown in paper I, by working in the full non-compact space rather than on compact sub-manifolds, one obtains an integral basis for the RR charges carried by the fractional two-branes. In examples with several divisors having non-trivial intersections, such as the resolution of $\mathbb{C}^3/\mathbb{Z}_5$, the restriction to a particular compact divisor is not useful either.

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As is well known, there is an intimate connection between the GLSM and toric geometry. We thus make use of standard constructions in toric geometry to systematise our study. We also provide a self-contained description of the necessary details in the sequel and the appendices. Based on these considerations, we write a general ansatz for the Chern character of the fractional two-branes and fix the coefficients using the open-string Witten index. However, this data is insufficient to fix all coefficients. We then provide additional input using the structure of the boundary states that uniquely fixes all coefficients. Finally, the heuristic method presented in I is used to provide candidate coherent sheaves that are compatible with computed local Chern characters.

2.1 Local Chern characters for sheaves on non-compact CY

In general, for a variety X, the Chern character is a map from the K-group $K(X)$ to the Chow group with rational coefficients $A^*(X) \otimes \mathbb{Q}$,

$$\mathrm{ch} : K_0(X) \to A^*(X) \otimes \mathbb{Q} \ . \tag{2.1}$$

By standard definitions we may identify

$$A^*(X) \otimes \mathbb{Q} = \oplus A^p(X) \otimes \mathbb{Q} = \oplus A_{n-p} \otimes \mathbb{Q} \ . \tag{2.2}$$

Following [15], it is particularly easy to write down the Chow group $A_k(X)$ in the case of a toric variety. For a toric variety $X = X(\Delta)$, where Δ is the corresponding fan, $A_k(X)$ is generated by all the classes of the orbit closures $V(\sigma)$ of $(n-k)$ dimensional cones of the fan Δ, modulo relations. With the last caveat on relations, the statement above is true for non-compact as well as compact varieties.

Now the orbit closures for a given fan Δ may be simply written down in terms of the divisors D_i, corresponding to the orbit closure of the one-dimensional cones, as well as the intersection of these divisors associated to orbit closures of higher dimensional cones spanned by these one-dimensional cones. Thus the Chern character involves in general a constant term for the rank, then terms of the form D_i, terms of the form $D_i \cdot D_j$ and then terms of the form $D_i \cdot D_j \cdot D_k$. There are no more terms since we are dealing with a complex threefold. For any given sheaf E, these terms have rational coefficients that have to be determined.

Note that in the case of a non-compact variety, some of these terms may be in fact zero. The classic example is the case of the the complex affine spaces, which have only one cohomology, so in fact the relations in the Chow group set all but one of the A_k to zero. However in our case without explicitly trying to determine such relations, we will use the fact that in the intersection form only such triple intersections will survive as are allowed by the toric construction. This we suppose will self-consistently determine which terms in the Chern character will be zero or non-zero. It is easy to see that similar computation for the affine spaces gives self-consistent results.

For completeness in appendix B, we give a brief introduction to some basic rules of toric geometry. We describe there how one can identify the various compact divisors from the toric data as well as the general rules for computing the triple intersections of divisors, which will be used in later sections.

2.2 Working with local Chern characters

After this prelude, we now move to the more practical aspects of working with local Chern characters. It was realised in [16] that the use of local Chern characters is necessary even for D-branes associated with compact submanifolds such as the fractional zero-branes. This is also true for our application where we consider fractional two-branes.

In this paper will be dealing exclusively non-compact manifolds X which are the crepant resolution of singular spaces $\mathbb{C}^3/\Gamma$, for some abelian discrete group $\Gamma \subset \mathrm{SU}(3)$. Let D_1, D_2 and D_3 denote the non-compact divisors and D_4, D_5, …denote the compact divisors that are added to resolve the singular space $\mathbb{C}^3/\Gamma$. These divisors thus correspond to four-cycles of X.[2] The double intersection of non-compact divisors give non-compact two-cycles and the other double intersections are compact two-cycles of X. If the triple intersection involves at least one compact divisor, then we get points with compact moduli spaces. Triple intersections that do not involve at least one compact divisor are set to zero.[3]

As already mentioned, one has linear equivalences amongst the divisors which lead to equivalences amongst the four- and two-cycles as well. In the non-compact situation, one has to be careful in using linear equivalences involving the non-compact and compact divisors. For instance, consider X to be the resolution of $\mathbb{C}^3/\mathbb{Z}_3$. The manifold X has one compact divisor, $D_4 = \mathbb{P}^2$. The linear equivalences are:

$$D_1 \sim D_2 \sim D_3 \ , \ \text{and} \quad D_1 + D_2 + D_3 + D_4 \sim 0 \ .$$

The second linear equivalence is valid *only* in the presence of a compact divisor. By this we mean that

$$D_4 \cdot (D_1 + D_2 + D_3 + D_4) \sim 0 \quad \text{but} \quad D_1 \cdot (D_1 + D_2 + D_3 + D_4) \nsim 0 \ .$$

With this caveat in mind, we can use the linear equivalences to simplify expressions.

The obvious inclusion maps for a (compact) divisor $j: \ D \to X$ can be used to push-forward the Chern characters of vector bundles on D to *local* Chern characters of sheaves on X. For instance, the push-forward of the structure sheaf $\mathcal{O}_D$ on X, denoted by $j_*(\mathcal{O}_D)$, is given by the sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to j_*(\mathcal{O}_D) \to 0 \tag{2.3}$$

which gives

$$\mathrm{ch}\big[j_*(\mathcal{O}_D)\big] = D - \frac{1}{2} D \cdot D + \frac{1}{6} D \cdot D \cdot D \ . \tag{2.4}$$

The local Chern character of all line-bundles are obtained by tensoring the above sequence suitably with an appropriate line-bundle. For vector bundles E with support on a divisor

[2] Poincaré duality provides an isomorphism between $H_4(X)$ and $H^2(X)$. Thus, when we write D_i in the Chern character, we take $D_i \in H^2(X)$.

[3] Mathematically speaking, it is well-known that the only non-zero triple intersections are the ones that involve at least one compact divisor. For an illustrative discussion of a closely related result in a related context see for instance the discussion in section 9, ch.2 of Iversen's text [17].

D the push-forward can be worked out by considering a resolution of E in terms of line bundles and then by pushing forward each term to X. The local Chern character of the push-forward sheaf, $j_*(E)$, can then be computed in a straightforward fashion. As an example, let D denote the compact divisor in the resolution of $\mathbb{C}^3/\mathbb{Z}_3$ and $\Omega_{\mathbb{P}^2}(1)$ the vector bundle given by the following exact sequence (the Euler sequence)

$$0 \to \Omega^1_{\mathbb{P}^2}(1) \to \mathcal{O}^{\oplus 3}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2} \to 0 \ . \tag{2.5}$$

The local Chern character of the push-forward of $\Omega^1_{\mathbb{P}^2}(1)$ is then given by

$$\mathrm{ch}\big[j_*(\Omega_{\mathbb{P}^2}(1))\big] = 3\ \mathrm{ch}\big[j_*(\mathcal{O}_{\mathbb{P}^2})\big] - \mathrm{ch}\big[j_*(\mathcal{O}_{\mathbb{P}^2}(1))\big] \ . \tag{2.6}$$

Before we write down an explicit form in specific cases for the Chern character it is useful also to take into account some simplification provided by linear equivalences and symmetry. Thus for the Chern character of a sheaf E on the blow-up of $\mathbb{C}^3/\mathbb{Z}_3$, we can write the following ansatz:

$$\mathrm{ch}(E) = a_1' + a_2 D_4 + a_2' D_1 + a_3 D_1 \cdot D_4 + a_3' D_1 \cdot D_2 + a_4 p\,, \tag{2.7}$$

where the prime indicates terms involving only non-compact divisors. Note that we have used the linear equivalence among the $D_i (i \neq 4)$, but have not used it for D_4. We remind the reader that the allowed triple intersection terms should have at least one D_4 in them. To save on notation we have clubbed all triple intersection terms together and re-written the term as a single coefficient times the class of a point p.

Obviously for the toric description of the blow-up of other $\mathbb{C}^3/\mathbb{Z}_n$ orbifolds the form of $\mathrm{ch}(E)$ will be different depending on the structure of the appropriate fan and the corresponding compact and non-compact divisors. We write them down for specific cases in the sequel.

3. Local Chern characters of the fractional two-branes: $\mathbb{C}^3/\mathbb{Z}_3$

3.1 General method

We now discuss the general method that we use to compute the local Chern character for the fractional two-branes. Unlike the fractional zero-branes where one needs to use the McKay correspondence to obtain the Chern character, for the fractional two-branes additional information is given by the open-string Witten index for strings connecting them to fractional zero-branes and as we will demonstrate will prove sufficient to fix the local Chern character of the fractional two-branes. The inputs that are used to compute the Chern character of the fractional two-branes are the following.

- The intersection matrix (this is called $\mathcal{I}^{0,2}$ in the sequel) which encodes the open-string Witten indices between the various fractional zero- and two-branes. This is taken from the CFT computation since it is independent of Kähler moduli.

- An ansatz for the local Chern character analogous to eq. (2.7) that takes into account linear equivalences among divisors.

- The non-compact terms in the local Chern character are identical for all fractional two-branes and equals $D_i \cdot D_j$ (for some i, j) when the fractional two-brane is given by $\phi_i = \phi_j = 0$. This corresponds to a specific choice for the coefficients associated with the non-compact divisors (we indicate these coefficients with a prime in our ansatz). This is justified later.

With these inputs, the local Chern character is *uniquely* fixed up to the class of a point. We will show that these Chern characters that we calculate provide a justification for the imprecise and heuristic way of accounting for the fractional charge that we had given in paper I.

We will now present two arguments, one geometric and the other a worldsheet one, to show that the non-compact contribution to the Chern character of the fractional two-brane is given by $D_i \cdot D_j$. The geometric argument is as follows. As a concrete example, consider the non-compact terms in the ansatz for $\mathbb{C}^3/\mathbb{Z}_3$ as given in eq. (2.7). A non-vanishing value for a'_1 corresponds to a (non-compact) $D6-brane$ wrapping the resolution of $\mathbb{C}^3/\mathbb{Z}_3$ while a non-vanishing value for a'_2 corresponds to a (non-compact) $D4$-brane wrapping the four-cycle given by say, $\phi_1 = 0$. Clearly, this cannot be the case since the large-volume limit corresponds to blowing up compact four-cycles. Thus we conclude $a'_1 = a'_2 = 0$. This argument is valid for other examples as well. For a fractional two-brane given by $\phi_i = \phi_j = 0$, again one can rule out all contributions other than $D_i \cdot D_j$. The second argument is a worldsheet one. Recall that the resolution of the orbifold singularity arise from closed-string moduli that appear in the twisted sector in the CFT. At the CFT point, the D-branes are described by boundary states which can be schematically written as

$$|B\rangle = |B\rangle_{\text{untwisted}} + |B\rangle_{\text{twisted}} \ . \tag{3.1}$$

where we have explicitly separated contributions from the untwisted sector and the twisted sector. The $|B\rangle_{\text{untwisted}}$ does not couple to the Kähler moduli. In particular, the one-point function on a disk of the corresponding vertex operator obtains a vanishing contribution from the untwisted sector. This also implies that the separation in eq. (3.1) holds even after resolving the singularity and in particular, in the large volume limit. Thus, one expects

$$|B\rangle^{\text{large volume}} = |B\rangle_{\text{untwisted}} + |B\rangle^{\text{deformed}}_{\text{twisted}} \ , \tag{3.2}$$

where the twisted sector boundary state is deformed by the Kähler moduli while the untwisted sector boundary state is identical to the CFT one. The non-compact terms arise solely from $|B\rangle_{\text{untwisted}}$ and are identical for all fractional two-branes and matches the expectation that it be $D_i \cdot D_j$ for a fractional brane given by the boundary condition $\phi_i = \phi_j = 0$ at the CFT end. This is also consistent with the geometrical picture of these fractional two-branes as extended objects not localised at the singularity.

3.2 Toric geometry of $\mathbb{C}^3/\mathbb{Z}_3$

The simplest orbifold to consider is $\mathbb{C}^3/\mathbb{Z}_3$ (and its unique crepant resolution) with $\mathbb{Z}_3$ action $\frac{1}{3}(1,1,1)$.[4] The resolution of the orbifold requires the blowing up of the singular

[4]The notation used here follows the one used in paper I.

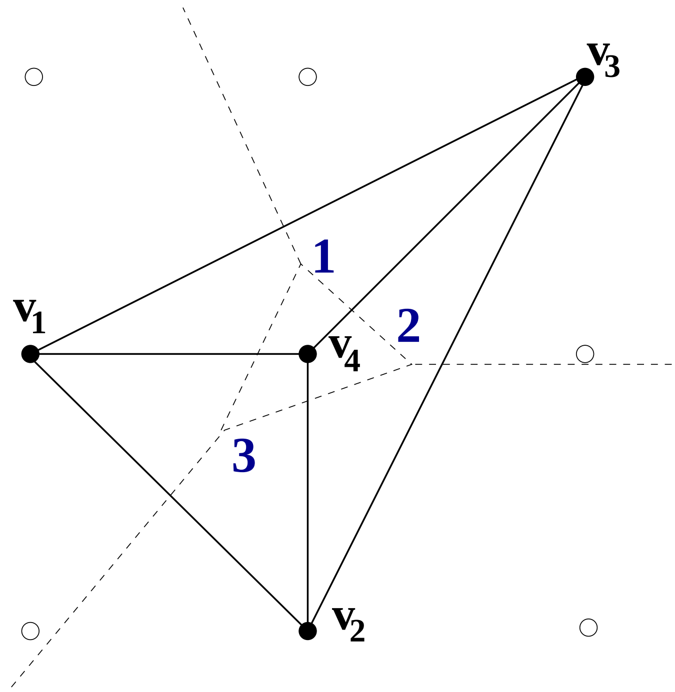

Figure 1: Toric diagram for $\mathbb{C}^3/\mathbb{Z}_3$. The dotted lines indicated the dual polytope.

point at the origin to a $\mathbb{P}^2$. The toric data associated with the orbifold is given by three vectors (see appendix B and ref. [18] for a review)

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}. \tag{3.3}$$

The crepant resolution of the orbifold is given by the addition of one vector:

$$v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.4}$$

The vector v_4 is associated with a compact divisor $D_4 = \mathbb{P}^2$. The four vectors are not independent and satisfy a relation, which we write as

$$\sum_{i=1}^{5} Q_i \; v_i = 0\,, \quad a = 1, 2$$

with

$$Q_i = \begin{pmatrix} 1 & 1 & 1 & -3 \end{pmatrix}$$

This toric data is represented by the figure given below, in which the various cones have been labelled as well.

The toric data can be naturally interpreted in terms of the Gauged Linear Sigma Model (GLSM) [19]. The GLSM associated with this toric data consists of four fields ϕ_i (one for each vector v_i) and one U(1) with charge vectors Q_i. The D-term equations are

$$|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 - 3|\phi_4|^2 = r \tag{3.5}$$

From the above D-term condition, we see that for $r \gg 0$ and $\phi_4 = 0$, we have a $\mathbb{P}^2$ with homogeneous coordinates ϕ_1, ϕ_2 and ϕ_3. The orbifold limit is obtained when $r \ll 0$. Here ϕ_4 necessarily has a non-zero vacuum expectation value $= \sqrt{|r|/3}$ and the U(1) is broken to a $\mathbb{Z}_3$ with an action of $\frac{1}{3}(111)$ on ϕ_1, ϕ_2 and ϕ_3 respectively. The divisors D_i are associated with the four-cycles given by $\phi_i = 0$.

3.3 Triple Intersections

The linear equivalences among the divisors are

$$D_1 {\sim} D_2 {\sim} D_3 \ \text{ and } D_1 + D_2 + D_3 + D_4 {\sim} 0 \ .$$

These equivalences are valid in the presence of a compact divisor. Intersections of the compact divisors among themselves are

$$D_4^3 = 9,\ D_4^2 \cdot D_1 = -3,\ D_4 \cdot D_1^2 = 1, \tag{3.6}$$

$$D_4 \cdot D_1 = h,\ D_4^2 = -3h \ . \tag{3.7}$$

From the above intersections we can write down the intersections of the compact and non-compact divisors with h

$$D_4 \cdot h = -3,\ D_1 \cdot h = 1 \ . \tag{3.8}$$

3.4 Fractional zero-branes

At the orbifold point, we impose Dirichlet boundary conditions, $\phi_i = 0$, $i = 1, 2, 3$. We get *three* fractional boundary states associated with these boundary conditions with a $\mathbb{Z}_3$ which cyclically permutes them. We label the analytic continuation of these D-branes to large-volume by $S_a^{(0)}$ $(a = 1, 2, 3)$. The open-string Witten index is invariant under this analytic continuation and provides an important input in our analysis [20]. At large-volume, it becomes the intersection form which we denote by $\langle E, F\rangle$ for two sheaves E and F. It is the defined by

$$\langle E, F\rangle = \int_X \mathrm{ch}(E^*)\ \mathrm{ch}(F)\ \mathrm{Td}(X) \ . \tag{3.9}$$

Define the matrix:

$$\mathcal{I}^{0,0}_{a,b} = \langle S_a^{(0)}, S_b^{(0)}\rangle \ . \tag{3.10}$$

Their intersection form computed as the open-string Witten index in the CFT is found to be

$$\mathcal{I}^{0,0} = -(1-g)^3 \,, \tag{3.11}$$

where g, the generator of the quantum $\mathbb{Z}_3$ at the orbifold point is given by the shift matrix

$$g = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} . \tag{3.12}$$

The Chern classes for the fractional zero-branes in this example is well known in the literature and were first determined in [21]. The Chern classes for the fractional zero-branes are known to be

$$\begin{aligned} \mathrm{ch}\big[S_0^{(0)}\big] &= D_4 + (3/2)h + (3/2)p = \mathrm{ch}\big[j_*(\mathcal{O}_{\mathbb{P}^2})\big] , \\ \mathrm{ch}\big[S_1^{(0)}\big] &= -2D_4 - 2h - p = -\mathrm{ch}\big[j_*(\Omega_{\mathbb{P}^2}(1))\big] , \\ \mathrm{ch}\big[S_2^{(0)}\big] &= D_4 + h/2 + p/2 = \mathrm{ch}\big[j_*(\mathcal{O}_{\mathbb{P}^2}(-1))\big] , \end{aligned} \tag{3.13}$$

where $j : D_4 \to X$. On can independently verify that the Chern classes written above are compatible with the CFT data.

3.5 Fractional two-branes

The fractional two-branes are obtained in the CFT by imposing Dirichlet boundary conditions $\phi_1 = \phi_2 = 0$ and imposing a Neumann boundary condition on ϕ_3. Using the general method in section 3.1, we are ready to determine the Chern characters of the large-volume analogues of the fractional two-branes. Define the matrices:

$$\begin{aligned} \mathcal{I}^{0,2}_{a,b} &= \langle S_a^{(0)}, S_b^{(2)} \rangle , \\ \mathcal{I}^{2,2}_{a,b} &= \langle S_a^{(2)}, S_b^{(2)} \rangle , \end{aligned} \tag{3.14}$$

where $S_a^{(0)}$ are the large volume analogues of the fractional zero-branes and the $S_b^{(2)}$ are the corresponding objects for the fractional two-branes.

From the CFT computations for the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold in I, we know that

$$\mathcal{I}^{0,2} = -(1-g)^2 \quad \text{and} \quad \mathcal{I}^{2,2} = g(1-g) . \tag{3.15}$$

It is now simple to insert ansatz (2.7) that we had written down for the Chern character of the $S_i^{(2)}$ above and try to solve for the coefficients using the CFT data given in eq. (3.15). The non-compact part of the Chern character is taken to be $D_1 \cdot D_2$, i.e., $a'_1 = a'_2 = 0$ and $a'_3 = 1$ for all three fractional two-branes. The Chern classes of the fractional two-branes as obtained from the orbifold intersection from is

$$\begin{aligned} \mathrm{ch}(S_0^{(2)}) &= D_4 + (3/2)h + D_1{\cdot}D_2 + a_4\ p , \\ \mathrm{ch}(S_1^{(2)}) &= -D_4 - (1/2)h + D_1{\cdot}D_2 + b_4\ p , \\ \mathrm{ch}(S_2^{(2)}) &= D_1{\cdot}D_2 + c_4\ p , \end{aligned} \tag{3.16}$$

where a_4, b_4, c_4 are the coefficients that are *not* fixed by the intersection numbers. Again, these choices are compatible with the second intersection form given in eq. (3.15). Note that $\mathcal{I}^{2,2}$ was not used in fixing the coefficients appearing in the ansatz (2.7) and thus it can be used as an additional check.

3.6 Geometry of the fractional two-branes

We can now see that these Chern characters provide some precise justification for the heuristic constructions of Paper I where we had constructed the large volume analogues of the fractional two-branes using some exact sequences (in particular, see section 4.5, eq. (4.20)-(4.22)). These sequences however were imprecisely defined, mathematically speaking, since we introduced fractional Chern classes for these objects. The meaning of these sequences now becomes clearer. The objects indicated by these sequences will be interpreted as contributing to the compact part of the Chern character, thus retaining their original mathematical meaning, while the fractional part will contribute precisely to the non-compact $D_1 \cdot D_2$ term of the Chern character. This (implicitly additive) separation is allowed since the Chern character is also additive for direct sums. We can rewrite eq. (3.8) as $D_4 \cdot D_4 \cdot D_2 = -3$ and $D_1 \cdot D_4 \cdot D_2 = 1$ utilising linear equivalence. In the background of a compact divisor, namely D_4, this reflects the fact that the non-compact contribution, $D_1 \cdot D_2$ accounts for exactly 1/3 of the unit charge given by the $D_4 \cdot D_2$ contribution. Thus using the eq. (4.26)-(4.28) of paper I we obtain the following:[5]

$$\begin{aligned} \mathrm{ch}\big[S_0^{(2)}\big] &= \mathrm{ch}\big[j_*(\mathcal{O}_{\mathbb{P}^2})\big] + D_1 \cdot D_2\,, \\ \mathrm{ch}\big[S_1^{(2)}\big] &= \mathrm{ch}\big[j_*(\mathcal{O}_{\mathbb{P}^2}(-1)\big] + D_1 \cdot D_2\,, \\ \mathrm{ch}\big[S_2^{(2)}\big] &= D_1 \cdot D_2\ . \end{aligned} \tag{3.17}$$

We can compute the Chern character of the pieces $\mathrm{ch}[j_*(\ldots)]$ using the techniques discussed in section 2 and verify that indeed we reproduce the Chern characters computed in the previous subsection in eq. (3.16). Thus the apparently imprecise sequences of paper I are indeed meaningful provided we exercise some care in interpretation.

We can now of course recall the physics side of the story from paper I in the language of what we had referred to as Higgs and Coulomb branes and their construction in terms of the bulk fermion degrees of freedom that survive on the boundary of the world-sheet. In order to do this, we first separate the contributions when $\phi_3 = 0$ from those that are present when $\phi_3 \neq 0$. At the orbifold point, the first contribution is localised at the singularity. This contribution is similar to that of fractional zero-branes on a $\mathbb{C}^2/\mathbb{Z}_3$ orbifold (where the coordinates of $\mathbb{C}^2$ are ϕ_1 and ϕ_2). Due to this similarity, following Martinec and Moore [29], we call the contribution when $\phi_3 = 0$ as the *Higgs branch* and the contribution when $\phi_3 \neq 0$ the *Coulomb branch*. As we will see, some of the fractional two-branes have both branches – we call them the *Higgs branes* while some only have a Coulomb branch — we call them the *Coulomb branes*.

We will now use a method that was alluded to but not used in paper I to extract the Higgs branch contributions of the fractional two branes. Now, there are only two fermions ξ_1 and ξ_2 coming from the Dirichlet boundary conditions[6] since the fermionic partners of the Neumann scalars do not contribute as suggested in paper I. The compact part of the

[5] An independent explanation is also given below.

[6] For Dirichlet boundary conditions preserving B-type supersymmetry, the ξ's are the linear combination *not* set to zero. Further, these are the "observables" in the topological B-model. See paper I for a more detailed discussion.

fractional two-branes are in one-to-one correspondence with the states given in the table below (the vacuum $|0\rangle$ satisfies $\bar{\xi}_i|0\rangle = 0$ for $i = 1, 2$) subject to the gaugino constraint being satisfied:

$$\phi_1\bar{\xi}_1 + \phi_2\bar{\xi}_2 = 0 \ . \tag{3.18}$$

Label	$S_0^{(2)}$	$S_1^{(2)}$	$S_2^{(2)}$
U(1) charge	0	1	2
State	$\|0\rangle$	$\xi_i\|0\rangle$	$\xi_1\xi_2\|0\rangle$
Higgs branch	D_4	D_4	—

In the above table, we indicate based on the detailed discussion below, the divisor on which the gaugino constraint can be satisfied when $\phi_3 = 0$. As explained earlier, we will distinguish the contribution when $\phi_3 = 0$ and when $\phi_3 \neq 0$ — both possibilities are permitted by the Neumann boundary condition. In the Higgs branch, at large volume, $\phi_1 = \phi_2 = 0$ is not allowed. This is however allowed in the Coulomb branch where $\phi_3 \neq 0$. Thus the Coulomb branch contribution to the Chern classes of the fractional two-branes contain $D_1 \cdot D_2$. This is consistent with the Chern character that we obtained for $S_2^{(2)}$ in eq. (3.16).

$S_1^{(2)}$ The gaugino constraint implies that we have a rank one bundle in the Higgs branch which can be identified with $j_*(\mathcal{O}_{\mathbb{P}^2}(-1))$.

$S_2^{(2)}$ This is a Coulomb brane since the gaugino constraint cannot be satisfied when $\phi_3 = 0$.

$S_0^{(2)}$ This is a line bundle which can be identified with $j_*(\mathcal{O}_{\mathbb{P}^2})$.

In this example, the Coulomb branch is identical to the non-compact contribution to the Chern class and is identical for all three fractional two-branes. As we will see, in the next example, the Coulomb branch is *not* the same as the non-compact term though it contains it. In the more intricate cases we will need both the precise Chern character computation and the physics construction of the boundary states using the fermions and their interpretation to pin down the objects corresponding to the large-volume analogues of all the fractional two-brane states

3.7 A change of basis

As we just saw, the Higgs branes have two kinds of contributions, one from the Higgs branch and the other from the Coulomb branch. We will now exhibit an *integral* change of basis which removes the Coulomb branch from the Higgs branes. Let $\hat{\mathbf{S}}^{(2)} = (\hat{S}_0^{(2)}, \hat{S}_1^{(2)}, \hat{S}_2^{(2)})^{\mathrm{T}}$ represent the new basis and $\mathbf{S}^{(2)}$ the original basis of fractional two-branes. Then,

$$\hat{\mathbf{S}}^{(2)} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{S}^{(2)} \ . \tag{3.19}$$

is the required change of basis. Note that it is an upper-triangular matrix.

4. The $\mathbb{C}^3/\mathbb{Z}_5$ example

We will consider the example of the orbifold $\mathbb{C}^3/\mathbb{Z}_5$ with the action $\frac{1}{5}(1,1,3)$. As discussed in appendix B, the resolution of this requires the blowing up of a $\mathbb{P}^2$ and a Hirzebruch surface $\mathbb{F}_3$ (which is a $\mathbb{P}^1$ fibration over $\mathbb{P}^1$) [16, 23]. The two exceptional divisors intersect along a curve which is a hyperplane on the $\mathbb{P}^2$. The toric data associated with the orbifold is given by three vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{4.1}$$

The unique crepant resolution of the orbifold is given by the addition of two vectors:

$$v_4 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \quad v_5 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.2}$$

As explained in appendix B, the vector v_4 is associated with the $\mathbb{P}^2$ and v_5 with the Hirzebruch surface $\mathbb{F}_3$. The five vectors are not independent and satisfy two relations, which we write as

$$\sum_{i=1}^{5} Q_i^A \; v_i = 0\,, \quad A = 1,2$$

with

$$Q_i^A = \begin{pmatrix} 1 & 1 & 0 & -3 & 1 \\ 0 & 0 & 1 & 1 & -2 \end{pmatrix} .$$

In figure 2, the toric data is represented by the following projection on a two-dimensional plane.

The GLSM associated with this toric data consists of five fields ϕ_i (one for each vector v_i) and two U(1)'s (one for each relation) with charge vectors Q_i^A. The D-term equations are

$$\begin{aligned} |\phi_1|^2 + |\phi_2|^2 + |\phi_5|^2 - 3|\phi_4|^2 &= r_1\,, \\ |\phi_3|^2 + |\phi_4|^2 - 2|\phi_5|^2 &= r_2\ . \end{aligned} \tag{4.3}$$

From the first D-term condition, we see that for $r_1 \gg 0$ and $\phi_4 = 0$, we have a $\mathbb{P}^2$ with homogeneous coordinates ϕ_1, ϕ_2 and ϕ_5. The base of $\mathbb{F}_3$ is the $\mathbb{P}^1$ is given by the hypersurface $\phi_5 = 0$ in the $\mathbb{P}^2$ and thus has homogeneous coordinates ϕ_1 and ϕ_2. The second D-term for $r_2 \gg 0$ and $\phi_5 = 0$, gives a $\mathbb{P}^1$ with homogeneous coordinates ϕ_3 and ϕ_4. This $\mathbb{P}^1$ is the fibre of $\mathbb{F}_3$.

The orbifold limit is obtained by first considering a particular linear combination of the above two D-term conditions, i.e., the one associated with $Q_i \equiv (Q_i^1 + 3Q_i^2)$:

$$|\phi_1|^2 + |\phi_2|^2 + 3|\phi_3|^2 - 5|\phi_5|^2 = r_1 + 3r_2\ . \tag{4.4}$$

In the limit $(r_1 + 3r_2) \ll 0$, ϕ_5 necessarily has a non-zero vev $= \sqrt{|r_1 + 3r_2|/5}$ and the associated U(1) is broken to a $\mathbb{Z}_5$ with an action of $\frac{1}{5}(113)$ on ϕ_1, ϕ_2 and ϕ_3 respectively.

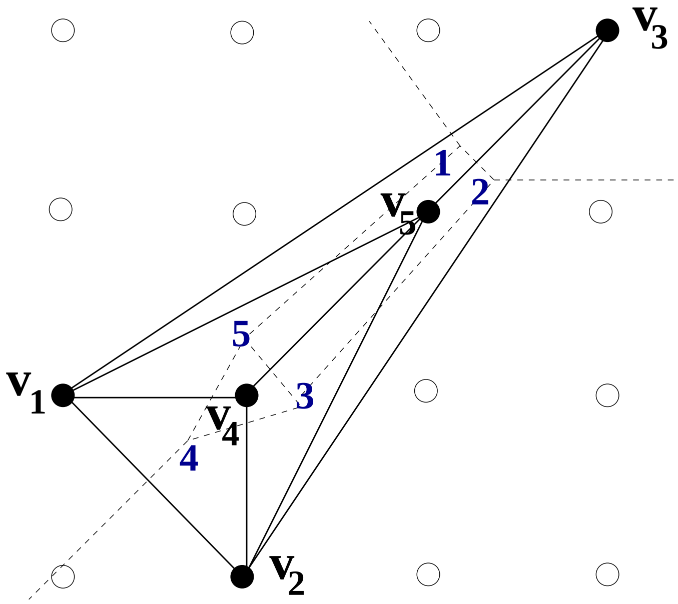

Figure 2: Toric diagram for $\mathbb{C}^3/\mathbb{Z}_5$

4.1 Triple Intersections

The linear equivalences among the divisors are

$$D_1{\sim}D_2{\sim}D_5+2D_3 \text{ and } D_1+D_2+D_3+D_4+D_5{\sim}0 \; .$$

These equivalences are valid in the presence of a compact divisor. Intersections of the compact divisors among themselves are

$$D_4^3=9,\; D_4^2{\cdot}D_5=-3,\; D_4{\cdot}D_5^2=1,\; D_5^3=8 \; . \tag{4.5}$$

Intersections of the compact divisors with the non-compact divisors are

$$\begin{aligned} D_4^2\cdot D_1=-3,\; D_4\cdot D_5\cdot D_1=1,\; D_5^2\cdot D_1=-2,\; D_5^2\cdot D_3=-5\,, \\ D_1^2\cdot D_4=1,\; D_5\cdot D_1^2=0,\; D_5\cdot D_1\cdot D_3=1,\; D_3^2{\cdot}D_5=3 \; . \end{aligned} \tag{4.6}$$

We also have

$$D_4\cdot D_5=D_4\cdot D_1=h,\; D_4\cdot D_3=0,\; D_5\cdot D_1=f,\; D_3\cdot D_5=h+3f \; . \tag{4.7}$$

where f is the $\mathbb{P}^1$ fibre of $\mathbb{F}_3$ and h is the hyperplane in $\mathbb{P}_2$. The self intersections of D_4 and D_5 are

$$D_4^2=-3h,\; D_5^2=-2h-5f \; .$$

From the above intersections we can write down the intersections of the compact and non-compact divisors with h and f

$$\begin{aligned} D_4{\cdot}h=-3,\; D_4{\cdot}f=1,\; D_5{\cdot}h=1,\; D_5{\cdot}f=-2\,, \\ D_1{\cdot}h=1,\; D_1{\cdot}f=0,\; D_3{\cdot}h=0,\; D_3{\cdot}f=1 \; . \end{aligned} \tag{4.8}$$

4.2 Fractional zero-branes

At the orbifold point, we impose Dirichlet boundary conditions, $\phi_i = 0$, $i = 1, 2, 3$. We get *five* fractional boundary states associated with these boundary conditions with a $\mathbb{Z}_5$ which cyclically permutes them. Their intersection form is

$$\mathcal{I}^{0,0} = -(1-g)^2(1-g^3)\,, \tag{4.9}$$

where g denotes the 5×5 shift-matrix that generates the $\mathbb{Z}_5$. One can choose an ansatz analogous to the one in (2.7) and determine the Chern characters of the fractional branes using the above intersection form. The relevant ansatz is

$$\mathrm{ch}(E) = a_1' + a_2 D_4 + a_3 D_5 + a_2' D_1 + a_3' D_3 + a_4 h + a_5 f + a_4' D_1^2 + a_5' D_1 \cdot D_3 + a_6 p\ . \tag{4.10}$$

Of course, the compact nature of the fractional zero-branes implies that all the non-compact pieces (indicated by adding a prime to the coefficient) vanish for fractional zero-branes. The fractional zero-branes are known to have the following local Chern character

$$\begin{aligned} \mathrm{ch}\big[S_0^{(0)}\big] &= D_4 + D_5 + (3/2)h + (5/2)f\,, \\ \mathrm{ch}\big[S_1^{(0)}\big] &= -2D_4 - D_5 - 2h - (3/2)f\,, \\ \mathrm{ch}\big[S_2^{(0)}\big] &= D_4 + h/2\,, \\ \mathrm{ch}\big[S_3^{(0)}\big] &= -D_5 - (5/2)f\,, \\ \mathrm{ch}\big[S_4^{(0)}\big] &= D_5 + (3/2)f\ . \end{aligned} \tag{4.11}$$

These were first obtained in [23, 16] using the McKay correspondence which we discuss in section 5 to obtain the Chern characters of the fractional zero-branes. This is done by first obtaining the tautological bundles and then computing their duals. The Chern character is clearly compatible with eq. (4.10) and one sees that the coefficients of the non-compact terms vanish as expected.

As this has not been discussed earlier in the literature, we will now write out concrete objects which will correspond to specific choices for the coefficient of the class of a point using the physical method proposed in paper I. The gaugino constraint (for the vector multiplet associated with the D-term in eq. (4.4)) is

$$\phi_1\bar{\xi}_1 + \phi_2\bar{\xi}_2 + 3\phi_3\bar{\xi}_3 = 0\ . \tag{4.12}$$

The fractional zero-branes are in one-to-one correspondence with the states [22, 14]: (the vacuum $|0\rangle$ satisfies $\bar{\xi}_i|0\rangle = 0$ and the index $a = 1, 2$) subject to the gaugino constraint eq. (4.12) being satisfied. In the third row of the above table, we indicate the divisors on which this is possible based on the following considerations. Let us consider the various states and identify the corresponding coherent sheaves.

$S_1^{(0)}$ The gaugino constraint is trivially satisfied when both $\phi_1 = \phi_2 = 0$. Thus, the sheaf has rank two when $\phi_1 = \phi_2 = 0$ and rank one when either $\phi_1 \neq 0$ and/or $\phi_2 \neq 0$. By studying the two D-terms given in eq. (4.3), we can see that the rank two condition

Label	$S_0^{(0)}$	$S_1^{(0)}$	$S_2^{(0)}$	$S_3^{(0)}$	$S_4^{(0)}$
U(1) charge	0	1	2	3	4
State	$\vert 0\rangle$	$\xi_a\vert 0\rangle$	$\xi_1\xi_2\vert 0\rangle$	$\xi_3\vert 0\rangle$	$\xi_a\xi_3\vert 0\rangle$
	D_4 and D_5	D_4 and D_5	D_4	D_5	D_5

is possible (at large volume) only if $\phi_5 \neq 0$. Thus, the rank two part has support on the compact divisor D_4 while the rank one part has support on D_5. Thus, we expect the following to hold:

$$S_1^{(0)} = i_*(V) + j_*(W)\ .$$

where V is a rank-two bundle on $D_4 = \mathbb{P}^2$ and W is a line-bundle on $D_5 = \mathbb{F}_3$. In fact, one can identify V with $\Omega_{\mathbb{P}^2}(1)$ and W with $\mathcal{O}_{D_5}(-D_1 - D_4)$.

$S_2^{(0)}$ The gaugino constraint holds only when $\phi_1 = \phi_2 = 0$ and hence S_2 has support on D_4 where it is a line-bundle $\mathcal{O}_{\mathbb{P}^2}(-1)$.

$S_3^{(0)}$ The gaugino constraint holds when $\phi_3 = 0$ which requires $\phi_4 \neq 0$. Thus, $S_3^{(0)}$ is the push-forward of a line-bundle on D_5. The line bundle can be identified with $\mathcal{O}_{D_5}(-D_4)$.

$S_4^{(0)}$ The gaugino constraint holds when $\phi_3 = 0$ and when either $\phi_1 \neq 0$ or $\phi_2 \neq 0$. The D-term constraints imply that S_4 is the push-forward of a line-bundle on D_5. The line bundle can be identified with $\mathcal{O}_{D_5}(-D_1 - D_4)$.

$S_0^{(0)}$ The gaugino constraint trivially holds and hence S_0 is the direct sum of the push-forward of line-bundles on D_4 and D_5. The two line bundles are $\mathcal{O}_{\mathbb{P}^2}$ and $\mathcal{O}_{D_5}(-D_4)$.

We will now move on to the fractional two-branes next. There two inequivalent types of fractional two-branes in this example — we can impose Neumann boundary condition on ϕ_1 or ϕ_3. We will label them A and B respectively instead of the notation $S^{(2)}$ used in the $\mathbb{C}^3/\mathbb{Z}_3$ example.

4.3 Fractional two-branes — Type I

We will first consider the fractional two-branes obtained by imposing a Neumann boundary condition on ϕ_1 and Dirichlet boundary conditions $\phi_2 = \phi_3 = 0$. The master formula given in paper I provides us the intersection form amongst the fractional two-branes as well as the intersection with the fractional zero-branes. We obtain

$$\mathcal{I}^{0,2} = -(1 + g - g^2 - g^4)\,, \tag{4.13}$$

$$\mathcal{I}^{2,2} = -(g^4 - g)\ . \tag{4.14}$$

The Chern classes of the fractional two-branes as obtained using the method discussed in section 3.1 are

$$\mathrm{ch}[A_0] = D_4 + D_5 + D_2{\cdot}D_3 + (3/2)h + (5/2)f\ ,$$

$$
\begin{aligned}
\mathrm{ch}[A_1] &= -D_4 + D_2{\cdot}D_3 - (1/2)h + f\,,\\
\mathrm{ch}[A_2] &= D_2{\cdot}D_3 + f\,,\\
\mathrm{ch}[A_3] &= -D_5 + D_2{\cdot}D_3 - (3/2)f\,,\\
\mathrm{ch}[A_4] &= D_2 \cdot D_3\ .
\end{aligned} \tag{4.15}
$$

The non-compact contribution is $D_2 \cdot D_3$ as discussed in section 3.1.

We now proceed to obtain coherent sheaves which reproduce the above Chern classes. For the chosen boundary condition, there are only two fermions ξ_2 and ξ_3. The Higgs branch of the fractional two-branes are in one-to-one correspondence with the states: (the vacuum $|0\rangle$ satisfies $\bar{\xi}_i|0\rangle = 0$ for $i = 2, 3$.) We will work out the component corresponding to the Higgs branch i.e., when $\phi_1 = 0$. For D_4, this picks out a $\mathbb{P}^1 \in D_4$ (with homogeneous coordinates ϕ_2 and ϕ_5) and a $\mathbb{P}^1 \in D_5$ (with homogeneous coordinates ϕ_3 and ϕ_4). subject

Label	A_0	A_1	A_2	A_3	A_4
U(1) charge	0	1	2	3	4
State	$\|0\rangle$	$\xi_2\|0\rangle$	–	$\xi_3\|0\rangle$	$\xi_2\xi_3\|0\rangle$
Higgs branch	D_4 and D_5	D_4	—	D_5	—

to the gaugino constraint eq. (4.12) being satisfied.

In the Higgs branch where $\phi_1 = 0$, at large volume, $\phi_2 = \phi_3 = 0$ is not allowed and $\phi_2 = \phi_5 = 0$ is also not allowed. These are however allowed in the Coulomb branch where $\phi_1 \neq 0$. Thus the Coulomb branch contributions to the Chern classes of the fractional two-branes contain $D_2 \cdot D_3$ and/or $D_2 \cdot D_5 = f$. This is consistent with the Chern classes that we obtain for A_2 and A_4 which have a vanishing Higgs branch. Clearly, we see that the Coulomb branch involves a compact piece as well.

A_1 The gaugino constraint needs $\phi_2 = 0$. Thus, one needs $\phi_5 \neq 0$. Thus, the fractional two-brane has support only on D_4 and is the line-bundle $\mathcal{O}_{D_4}$.

A_2 This is one of the Coulomb branes.

A_3 This needs $\phi_3 = 0$. It has support on D_5 alone and the Higgs branch is the line bundle $\mathcal{O}_{D_5}(-D_1 - D_4)$.

A_4 This needs $\phi_2 = \phi_3 = 0$. So this is another Coulomb brane.

A_0 This has support on both D_4 and D_5 and the Higgs branch is the push-forward of two line bundles on these divisors. These are $\mathcal{O}_{D_4}$ and $\mathcal{O}_{D_5}(-D_4)$ respectively.

Comment: A_0 A_1, A_2 are in some ways similar to fractional two-branes on the resolution of $\mathbb{C}^3/\mathbb{Z}_3$.

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4.4 Fractional two-branes — Type II

We will consider the fractional two-branes obtained by imposing Neumann boundary conditions on ϕ_3 and $\phi_1 = \phi_2 = 0$. The master formula given in paper I provides us the intersection form amongst the fractional two-branes as well as the intersection with the fractional zero-branes. We obtain

$$\mathcal{I}^{0,2} = -g^2(1-g)^2 \,, \tag{4.16}$$
$$\mathcal{I}^{2,2} = g - g^2 + g^3 - g^4 \ . \tag{4.17}$$

The Chern classes obtained from the intersection form at the orbifold point are (ignoring the class of a point)

$$\begin{aligned} \mathrm{ch}[B_0] &= D_4 + D_5 + D_1 \cdot D_2 + (3/2)h + (5/2)f \,, \\ \mathrm{ch}[B_1] &= -D_4 - D_5 + D_1 \cdot D_2 - h/2 - (3/2)f \,, \\ \mathrm{ch}[B_2] &= D_1 \cdot D_2 \,, \\ \mathrm{ch}[B_3] &= D_4 + D_1 \cdot D_2 + (3/2)h \,, \\ \mathrm{ch}[B_4] &= -D_4 + D_1 \cdot D_2 - h/2 \ . \end{aligned} \tag{4.18}$$

For the given boundary condition, there are only two fermions ξ_1 and ξ_2. The Higgs branch of the fractional two-branes are in one-to-one correspondence with the states: (the vacuum $|0\rangle$ satisfies $\bar{\xi}_a|0\rangle = 0$ for $a = 1, 2$.) subject to the gaugino constraint eq. (4.12)

Label	B_0	B_1	B_2	B_3	B_4
U(1) charge	0	1	2	3	4
State	$\|0\rangle$	$\xi_a\|0\rangle$	$\xi_1\xi_2\|0\rangle$	—	—
Higgs branch	D_4 and D_5	D_5	—	—	—

being satisfied.

The Higgs branch is when $\phi_3 = 0$. In this case, at large volume, it is **not** possible to have either $\phi_4 = 0$ or $\phi_1 = \phi_2 = 0$. This is possible in the Coulomb branch where $\phi_3 \neq 0$. Thus the contributions of the Coulomb branes can arise these two sources. The associated Chern classes are $(D_4 + 3h/2 + 3p/2)$ (from $\phi_4 = 0$) and $D_1 \cdot D_2$ (from $\phi_1 = \phi_2 = 0$).

B_1 The contribution that arises on D_5 is as in the case of $S_1^{(0)}$ and is the line-bundle $\mathcal{O}_{D_5}(-D_1 - D_4)$. The D_4 appearing in the Chern character must come from the Coulomb branch since $\phi_4 = 0$ is not allowed at large volume when $\phi_3 = 0$.

B_2 This has support when $\phi_1 = \phi_2 = 0$. In the Higgs branch, $\phi_3 = 0$ and hence this is not allowed. Hence, this is a Coulomb brane.

B_3 This is one of the Coulomb branes.

B_4 This is one of the Coulomb branes.

B_0 The discussion is similar to $S_0^{(0)}$ and the Higgs branch of the sheaf has support on both D_4 and D_5 and can be identified with the direct sum of the push-forward of the line-bundles $\mathcal{O}_{\mathbb{P}^2}$ and $\mathcal{O}_{D_5}(-D_4)$.

4.5 A change of basis

As we just saw, the Higgs branes have two kinds of contributions, one from the Higgs branch and the other from the Coulomb branch. We will now exhibit an *integral* change of basis which removes the Coulomb branch from the Higgs branes. Let $\hat{\mathbf{A}} = (\hat{A}_0, \ldots, \hat{A}_4)^{\mathrm{T}}$ represent the new basis for type I fractional two-branes and $\hat{\mathbf{B}}$ the new basis for the type II fractional two-branes. Then,

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{A}\,, \tag{4.19}$$

$$\hat{\mathbf{B}} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{B}\,. \tag{4.20}$$

are the required change of bases. It is of interest if this change of basis is related to a change of basis proposed by Moore and Parnachev in [24].

5. Quantum McKay correspondence

The McKay correspondence [25 – 27] can be stated in several different forms. We will consider the one due to Ito and Nakajima [28] where the McKay correspondence is presented as a duality between two families of sheaves. The first family is given by the coherent sheaves associated with the fractional zero-branes and the second one is associated with the so-called *tautological* bundles. The duality is stated as [16]

$$(S_a^{(0)}, R^b_{(0)})_X \equiv \int_X \mathrm{ch}(R^b_{(0)})\ \mathrm{ch}(S_a^{(0)})\ \mathrm{Td}(X) = \delta_a^b\,, \tag{5.1}$$

where we remind the reader that X is the crepant resolution of the orbifold. Note that the above expression is not the intersection form and hence we indicate the inner product by $(,)$ rather than $\langle,\rangle$. Inspired by this, a generalisation called the quantum McKay correspondence was proposed in I for the fractional $2p$ -branes and is stated as the following duality:

$$(S_a^{(2p)}, R^b_{(2p)})_X \equiv \int_X \mathrm{ch}(R^b_{(2p)})\ \mathrm{ch}(S_a^{(2p)})\ \mathrm{Td}(X) = \delta_a^b\,. \tag{5.2}$$

This seems to be related to a correspondence of Martinec and Moore [29] in the context of non-supersymmetric orbifolds. In the following we shall obtain the Chern characters of the duals for the fractional two-branes in the two working examples in the paper: $\mathbb{C}^3/\mathbb{Z}_3$, $\mathbb{C}^3/\mathbb{Z}_5$. As always, the intersection numbers do not uniquely fix the Chern class due to the linear equivalences among the divisors. Unlike the fractional two-branes, we are unable to fix this ambiguity by appealing to CFT.

5.1 $\mathbb{C}^3/\mathbb{Z}_3$ orbifold

The Chern characters for the R-sheaves corresponding to the Higgs branes are

$$\begin{aligned} \mathrm{ch}(R^0_{(2)}) &= D_1 - \frac{D_1^2}{2}\,, \\ \mathrm{ch}(R^1_{(2)}) &= D_1 - \frac{3D_1^2}{2}\,, \end{aligned} \tag{5.3}$$

and for the R-sheaf dual to the Coulomb brane among the fractional two-branes, the Chern character is

$$\mathrm{ch}(R^2_{(2)}) = D_1 + D_4 - \frac{2h}{3}\ . \tag{5.4}$$

5.2 $\mathbb{C}^3/\mathbb{Z}_5$ orbifold

Recall that there are two types of fractional two-branes in this example. We quote the duals for both types of fractional two-branes.

Type I. The Chern characters of the R-sheaves dual to the Higgs branes are:

$$\begin{aligned} \mathrm{ch}(R^0_{(2)}) &= D_1 - \frac{D_1^2}{2}\,, \\ \mathrm{ch}(R^1_{(2)}) &= D_1 - \frac{3D_1^2}{2}\,, \\ \mathrm{ch}(R^3_{(2)}) &= D_1 - \frac{D_1^2}{2} - D_1{\cdot}D_3\ . \end{aligned} \tag{5.5}$$

The corresponding Chern characters for the R-sheaves dual the Coulomb branes with are:

$$\begin{aligned} \mathrm{ch}(R^2_{(2)}) &= D_1 + D_4 - \frac{5D_1^2}{2} - \frac{3h}{2}\,, \\ \mathrm{ch}(R^4_{(2)}) &= D_1 + D_4 + D_5 + \frac{D_1^2}{2} - \frac{h}{2} - \frac{3f}{2}\ . \end{aligned} \tag{5.6}$$

Type II. The Chern characters of the R-sheaves for the Higgs branes are:

$$\begin{aligned} \mathrm{ch}(R^0_{(2)}) &= D_3 - \frac{D_3^2}{2}\,, \\ \mathrm{ch}(R^1_{(2)}) &= D_3 - \frac{3D_3^2}{2} - D_3{\cdot}D_1\ . \end{aligned} \tag{5.7}$$

The Chern characters for the R-sheaves for the Coulomb branes are:

$$\begin{aligned} \mathrm{ch}(R^2_{(2)}) &= D_3 + D_4 + D_5 + \frac{D_3^2}{2} - \frac{h}{2} + \frac{f}{2}\,, \\ \mathrm{ch}(R^3_{(2)}) &= D_3 + D_5 - \frac{3D_3^2}{2} - h - \frac{7f}{2}\,, \end{aligned}$$

$$\mathrm{ch}(R^4_{(2)}) = D_3 + D_5 - h - \frac{9f}{2} - D_1{\cdot}D_3 - \frac{3D_3^2}{2}\ . \tag{5.8}$$

In the above expressions the class of a point has not been shown, as it is undetermined. We re-emphasise that in the above expressions, the terms corresponding to the fibres are fixed only up to linear equivalences, while the leading terms corresponding to the divisors are uniquely fixed by the equations.

Now one can try to write down explicit objects which have these Chern characters. Of course, we don't have the exact expressions for the complete Chern character. In particular the class of a point is undetermined, so we cannot hope to retrieve the explicit objects. However, the R-sheaves corresponding to the Higgs branes are expected to be given by line bundles with support on the appropriate non-compact divisor. Looking for line bundles as the corresponding objects, one can show that one can uniquely write down such line bundles such that their Chern characters match with expressions for the R-sheaves corresponding to the Higgs branes, up to the class of the point. The objects so obtained have a nice and simple structure. The explicit representations for the Higgs branch branes are:

For the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold:

$$R^0_{(2)} = i_*\left(\mathcal{O}_{D_1}\right)\ , \tag{5.9}$$
$$R^1_{(2)} = i_*\left(\mathcal{O}_{D_1}(-D_1)\right)\ , \tag{5.10}$$

where $i : D_1 \to X$.

The $\mathbb{C}^3/\mathbb{Z}_5$ orbifold

Type I.

$$R^0_{(2)} = i_*\left(\mathcal{O}_{D_1}\right)\ , \tag{5.11}$$
$$R^1_{(2)} = i_*\left(\mathcal{O}_{D_1}(-D_1)\right)\ , \tag{5.12}$$
$$R^3_{(2)} = i_*\left(\mathcal{O}_{D_1}(-D_3)\ ,\right. \tag{5.13}$$

where $i : D_1 \to X$.

Type II.

$$R^0_{(2)} = j_*\left(\mathcal{O}_{D_3}\right)\ , \tag{5.14}$$
$$R^1_{(2)} = j_*\left(\mathcal{O}_{D_3}(-D_1)\right)\ , \tag{5.15}$$

where $j : D_3 \to X$. Using certain sequences one can check that the Chern characters of these objects match with the expressions given above, up to the class of the point. This has been carried in the appendix C.

Now for the Coulomb branch branes we have no such guide to write down the objects, since there is no reason why they should be line bundles, for instance. Moreover, even if one assumes that they are line bundles the choice of object is not unique. This is simply because of the technical fact that there are more terms in the various expression of the Chern character and there are many possible ways in which they can be written.

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However the crucial point to note is that the structure of the Chern characters for these Coulomb branch branes is such that they cannot be written as objects restricted to the appropriate non-compact divisor. So they are in general of the form

$$R^i_{(2)} = i_* A + j_* B \,, \tag{5.16}$$

where A is an object with support on the corresponding non-compact divisor while B has support on a compact divisor.

In the appendix D, we will nevertheless write down simple representative objects for these Coulomb branch branes as well. It should be stated that this is not a unique representation.

6. Conclusion

In this paper we have continued further with the study of the quantum McKay correspondence that was proposed in our previous paper. To summarise, we have studied the fractional two-branes in the $\mathbb{C}^3/\mathbb{Z}_3$ and the $\mathbb{C}^3/\mathbb{Z}_5$ examples in the framework of toric geometry.

We have identified the Higgs branch branes as well as the Coulomb branch branes in these examples. We have discussed further the quantum McKay correspondence for the fractional 2 branes, generalising the McKay correspondence for the fractional zero branes.

We have given the explicit objects for the tautological branes corresponding to the Higgs branch branes in terms of line bundles, with support on appropriate non-compact divisors. For the Coulomb branch branes we see the associated R-sheaves are objects which cannot be written as objects with support only on the non-compact divisors.

Our analysis has been based on several choices we made in solving the related equations for computing the Chern character of the fractional two-branes as well as the R-sheaves. However we feel that the very fact that a solution exists with the desired features is non-trivial. A deeper understanding, both mathematically and physically, of the quantum McKay correspondence is desirable.

It would be interesting if a more direct relation of our analysis could be found to the discussion of Martinec and Moore using some version of the Hori-Vafa map [30].

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A. Some details of the push-forward

In this appendix, we will explain some of the relevant details of the push-forward map that has been used in the main text of the paper. Let us consider the map $i : X \to Y$. The push-forward in homology is straightforward — $i_* : H_k(X) \to H_k(Y)$. After all a k-cycle remains a k-cycle whether one is on one manifold or the other. What is non-trivial is the push-forward in cohomology which one can figure out by using the usual Poincaré duality. One first takes the Poincaré dual of a cohomology class in X, pushes forward the homology class by i_* and then takes the Poincaré dual again. If the dimensions of X and Y are m and n respectively, then $i_* : H^k(X) \to H^{n-m+k}(Y)$. The next important fact is to note that

$$\int_X \omega = \int_Y i_* \omega \ . \tag{A.1}$$

Notice that this works only if the entire integrand in the integral over Y is the i_* of something on X. To achieve this in general we need to use relations that are known in the mathematical literature as projection formula. (Such projection formulae are quite important and one needs to use the right one in context). The one relevant to us is as follows:

$$i_*(E \otimes i^* F) = i_*(E) \otimes F \, , \tag{A.2}$$

where $E \in H^*(X)$ and $F \in H^*(Y)$.

By the Grothendieck-Riemann-Roch theorem we have that

$$i_*\big[\mathrm{ch}(E)\ \mathrm{Td}(X)\big] = i_*\big[\mathrm{ch}(E)\big]\ \mathrm{Td}(Y) \ . \tag{A.3}$$

where $E \in H^*(X)$. Using the GRR theorem and the projection formula, we can show the intersection forms when computed in terms of local Chern characters in the total space are the same as the ones computed directly on the compact divisor.

B. Toric geometry - basics

In this appendix, we will briefly review how to construct toric diagrams for orbifolds as well as to read off various information about the orbifold space from the toric data. We will discuss the specific examples of $\mathbb{C}^3/\mathbb{Z}_3$,$\mathbb{C}^3/\mathbb{Z}_5$ orbifolds, which are discussed in the paper.

B.1 The $\mathbb{C}^3/\mathbb{Z}_3$ orbifold

First consider the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold with orbifold action $\frac{1}{3}[1,1,1]$. In the toric geometry picture this orbifold is represented by the cone spanned by the vertices

$$v_1 = (1,0,0),\ \ v_2 = (0,1,0),\ \ v_3 = (-1,-1,3) \ . \tag{B.1}$$

To see that this cone describes the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold, we first construct the dual cone. This is done by the following procedure. If (a,b,c) is a vector in the dual cone, then we look for those vectors such that the inner product of this with each of the above vertices is positive semidefinite. this gives the following inequalities.

$$a \geq 0,\ \ b \geq 0,\ \ 3c \geq b + a \ . \tag{B.2}$$

Now we have to solve these inequalities to get the basis vectors of the dual cone. All other solutions to (a, b, c) can be written as a positive linear combination of these basis vectors and moreover no basis vector can be expressed as a positive linear combination of any others. For the example at hand the solutions are given by the following 10 vectors.

$$\begin{aligned}
&v'_1 = (0,0,1),\ v'_2 = (3,0,1),\ v'_3 = (0,3,1),\ v'_4 = (2,1,1),\\
&v'_5 = (1,2,1),\ v'_6 = (1,1,1),\ v'_7 = (2,0,1),\ v'_8 = (0,2,1),\\
&v'_9 = (1,0,1),\ v'_{10} = (0,1,1)\ .
\end{aligned} \tag{B.3}$$

Each of these vectors is associated with a monomial. For example

$$v'_1 \equiv Z,\ v'_2 \equiv X^3 Z\ . \tag{B.4}$$

Now we will digress a bit. Consider polynomials in two variables (U, V). Then the domain over which these arbitrary polynomials are well defined ,which we denote by $\mathbb{C}[U, V]$, is actually $\mathbb{C}^2$, so $\mathbb{C}[U, V]$ is the coordinate ring of $\mathbb{C}^2$. We will use the shorthand notation $\mathbb{C}[U, V] \equiv \mathbb{C}^2$. Similarly if we look at the domain over which polynomials of the variables (U, V, U^{-1}, V^{-1}) are well defined it describes the space $(\mathbb{C}^*)^2$, because the functions are not defined at $(U, V) = (0, 0)$. Similarly if we consider polynomials in three variables (U, V, W), then $\mathbb{C}[U, V, W] \equiv \mathbb{C}^3$. The orbifold $\mathbb{C}^3/\mathbb{Z}_3$ with orbifold action $\frac{1}{3}$[1,1,1] on (U, V, W) can be described as the domain over which all polynomials constructed out of variables, which are single valued on the orbifold, is defined. Therefore,

$$\mathbb{C}^3/\mathbb{Z}_3 \equiv \mathbb{C}[U^3, V^3, W^3, UVW, UV^2, VU^2, VW^2, WV^2, UW^2, WU^2]\ . \tag{B.5}$$

Now we can see how to read off the space from the data we obtained from the dual cone. Writing the monomial associated to each of the dual basis vectors we construct the domain over which polynomials with these monomials as the variables are well defined. This in our notation is written as

$$\mathbb{C}[Z, X^3Z, Y^3Z, X^2YZ, XY^2Z, XYZ, X^2Z, Y^2Z, XZ, YZ]\ .$$

After changing variables to $X = \frac{U}{W}$, $Y = \frac{V}{W}$ and $Z = W^3$, we get

$$\mathbb{C}[W^3, U^3, V^3, U^2V, V^2U, UVW, U^2W, V^2W, UW^2, VW^2]\ .$$

This is the description of $\mathbb{C}^3/\mathbb{Z}_3$, that we saw earlier.

B.1.1 Resolution of the orbifold

To resolve the orbifold, the strategy is to subdivide the cone into several smaller cones by inserting more vectors in the interior of the cone such that for each sub-cone the determinant of the generators of that particular cone, which is also the volume of the particular cone, is one. One can easily see that this criteria is not satisfied by the original cone itself. For the the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold, this is achieved by taking one more vector

$$v_4 = (0, 0, 1)\,, \tag{B.6}$$

Cone 1	Cone 2	Cone 3
$c \geq 0,\ a \geq 0,\ 3c \geq a+b$	$c \geq 0,\ b \geq 0,\ 3c \geq a+b$	$c \geq 0,\ a \geq 0,\ b \geq 0$
$v_1' = (0,3,1)$,	$v_1' = (3,0,1)$,	$v_1' = (1,0,0)$,
$v_2' = (0,-1,0)$,	$v_2' = (-1,0,0)$,	$v_2' = (0,1,0)$,
$v_3' = (1,-1,0)$	$v_3' = (-1,1,0)$	$v_3' = (0,0,1)$
$\mathbb{C}[Y^3Z, Y^{-1}, XY^{-1}]$	$\mathbb{C}[X^3Z, X^{-1}, YX^{-1}]$	$\mathbb{C}[X,Y,Z]$

which subdivides the cone to three sub-cones, each of which have a unit determinant. The new cone so obtained is given in figure 1 Now as before construct the dual cones for each of the cones, and as before we have the following inequalities. The divisor corresponding to v_4 is given by $Z = 0$ and is obtained by substituting $Z = 0$ in the above. Then one has the following spaces $\mathbb{C}[X,Y]$, $\mathbb{C}[X^{-1}, YX^{-1}]$, $\mathbb{C}[Y^{-1}, XY^{-1}]$. These are to be thought of as local coordinate patches of some space. What space do these patches describe? They describe the space $\mathbb{P}^2$. This can be seen by looking at the patches of $\mathbb{P}^2$. $\mathbb{P}^2$ is given by $(U,V,W) \sim (\lambda U, \lambda V, \lambda W)$. Then we have three patches given by the regions where U,V,W are individually non zero. In each of these patches the coordinates can be taken to be $(\frac{V}{U}, \frac{W}{U})$, $(\frac{U}{V}, \frac{W}{V})$, $(\frac{U}{W}, \frac{V}{W})$. Defining $X = \frac{U}{W}$ and $Y = \frac{V}{W}$, we have the following three patches (X,Y), (X^{-1}, YX^{-1}), (Y^{-1}, XY^{-1}), so comparing with what we got from the toric analysis we see that the space after resolution is indeed a $\mathbb{P}^2$ so we see that $\mathbf{D_4} \equiv \mathbb{P}^\mathbf{2}$.

B.2 The $\mathbb{C}^3/\mathbb{Z}_5$ orbifold

Now we consider the example of the $\mathbb{C}^3/\mathbb{Z}_5$ orbifold with orbifold action $\frac{1}{5}$[1,1,3]. The vertices for the cone are given by

$$v_1 = (1,0,0),\ v_2 = (-1-3,5),\ v_3 = (0,1,0)\ . \tag{B.7}$$

Using the same method outlined before one can check that this is indeed the $\mathbb{C}^3/\mathbb{Z}_5$ orbifold.

B.2.1 The resolution of $\mathbb{C}^3/\mathbb{Z}_5$

Following the process for resolution as described in the $\mathbb{C}^3/\mathbb{Z}_3$ example one finds that one has to insert two vertices

$$v_4 = (0,-1,2),\ v_5 = (0,0,1), \qquad \text{see figure 2} \tag{B.8}$$

inside the cone to get the desired condition of unit determinant for the individual sub-cones. Now as before construct the dual fans for each of the cones, and as before we have the following inequalities.

$$\text{Cone 1: } c \geq 0,\ a \geq 0 \text{ and } b \geq 0\ . \tag{B.9}$$

The corresponding vertices are given by

$$v_1' = (1,0,0),\ v_2' = (0,1,0),\ v_3' = (0,0,1)\ . \tag{B.10}$$

The space is given by $\mathbb{C}[X, Y, Z]$

$$\text{Cone 2: } c \geq 0,\ 5c \geq 3b + a \text{ and } b \geq 0\ . \tag{B.11}$$

The corresponding vertices are given by

$$v_1' = (-1, 0, 0),\ v_2' = (-3, 1, 0),\ v_3' = (5, 0, 1)\ . \tag{B.12}$$

The space is given by $\mathbb{C}[X^{-1}, X^{-3}Y, X^5Z]$.

$$\text{Cone 3: } c \geq 0,\ 2c \geq b \text{ and } 5c \geq 3b + a\ . \tag{B.13}$$

The corresponding vertices are given by

$$v_1' = (3, -1, 0), v_2' = (-1, 2, 1), v_3' = (-1, 0, 0)\ . \tag{B.14}$$

The space is given by $\mathbb{C}[X^3Y^{-1}, Y^2ZX^{-1}, X^{-1}]$.

$$\text{Cone 4: } 2c \geq b,\ a \geq 0 \text{ and } 5c \geq 3b + a\ . \tag{B.15}$$

The corresponding vertices are given by

$$v_1' = (0, 5, 3),\ v_2' = (1, -2, -1),\ v_3' = (0, -2, -1)\ . \tag{B.16}$$

The space is given by $\mathbb{C}[Y^5Z^3, XY^{-2}Z^{-1}, Y^{-2}Z^{-1}]$.

$$\text{Cone 5: } c \geq 0, 2c \geq b \text{ and } a \geq 0\ . \tag{B.17}$$

The corresponding vertices are given by

$$v_1' = (0, -1, 0),\ v_2' = (1, 0, 0),\ v_3' = (0, 2, 1)\ . \tag{B.18}$$

The space is given by $\mathbb{C}[Y^{-1}, X, Y^2Z]$.

Now the divisor D_4 corresponding to v_4 is given by $\mathrm{D}_4 \equiv Z^2/Y = 0$. To find out what space this divisor corresponds to one has to analyse all the cones of which this is a common point. These will be the coordinate patches of the corresponding space. Since there are three cones surrounding this point, the corresponding space should be a $\mathbb{P}^2$. This can be checked rigorously, exactly as before. To do this we substitute $\mathrm{D}_4 = 0$ in the cones (3), (4) and (5). Take $Y^2Z \equiv A$ and $\mathrm{D}_4 = 0$. We then get for the corresponding patches, $\mathbb{C}[AX^{-1}, X^{-1}]$, $\mathbb{C}[A, X]$ and $\mathbb{C}[XA^{-1}, A^{-1}]$ respectively. As noted earlier these are the patches of $\mathbb{P}^2$, so

$$\mathrm{D}_4 \equiv \mathbb{P}^2\ . \tag{B.19}$$

To find the space corresponding to D_5 given by $Z = 0$ we similarly substitute $Z = 0$ in the patches (1),(2),(3),(5). We then get for the corresponding patches:

$$(1)\ \mathbb{C}[X, Y],\ (2)\ \mathbb{C}[X^{-1}, X^{-3}Y],\ (3)\ \mathbb{C}[X^3Y^{-1}, X^{-1}],\ (5)\ \mathbb{C}[Y^{-1}, X] \tag{B.20}$$

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These are the coordinate patches associated with the space $\mathbb{F}_3$. In general the space $\mathbb{F}_a$ has the following four coordinate patches [15].

$$(X^{-1}, X^a Y), (X, Y), (X^{-1}, X^{-a}Y^{-1}), (X, Y^{-1})$$

So we see that

$$\mathrm{D}_5 \equiv \mathbb{F}_3 \ . \tag{B.21}$$

The general rule for computing the triple intersections is that the triple intersections for any three distinct divisors which span a cone is 1 and the intersection of those distinct divisors that don't span a cone, vanish. For instance for the $\mathbb{C}^3/\mathbb{Z}_5$ orbifold, the vectors v_1, v_2, v_4 span the cone labelled 4. So we have $D_1 \cdot D_4 \cdot D_2 = 1$. On the other hand since v_1, v_2, v_5 do not span a cone, we have $D_1 \cdot D_5 \cdot D_2 = 0$, and so on. These triple intersections, involving the distinct divisors, can then be used to obtain other triple intersections, which involve self intersections of divisors. This is done by using the linear equivalence relations between the divisors to express these in terms of the ones involving distinct ones.

C. The R-sheaves for the Higgs branes

We will provide some details that go into computing the Chern character for the R-sheaves.[7] The general sequence we will be using is of the form,

$$0 \to \mathcal{O}_X(-D_i - D_j) \to \mathcal{O}_X(-D_j) \to i_*\left(\mathcal{O}_{D_i}(-D_j)\right) \to 0 \tag{C.1}$$

where $i : D_i \to X$. From the above sequence we obtain

$$\mathrm{ch}\left[i_*\left(\mathcal{O}_{D_i}(-D_j)\right)\right] = \mathrm{ch}\left[\mathcal{O}_X(-D_j)\right] - \mathrm{ch}\left[\mathcal{O}_X(-D_i - D_j)\right] \tag{C.2}$$

The R-sheaves for the Higgs branes of the $\mathbb{C}_3/\mathbb{Z}_3$ orbifold

$$0 \to \mathcal{O}_X(-D_1) \to \mathcal{O}_X \to i_*\left(\mathcal{O}_{D_1}\right) \to 0 \tag{C.3}$$

Using the above sequence can show that,

$$\mathrm{ch}\left[i_*\left(\mathcal{O}_{D_1}\right)\right] = \mathrm{ch}\left[R_2^0\right], \tag{C.4}$$

up to the class of a point, which was undetermined in the main text. Similarly for the other R-sheaf, consider the following sequence

$$0 \to \mathcal{O}_X(-2D_1) \to \mathcal{O}_X(-D_1) \to i_*\left(\mathcal{O}_{D_1}(-D_1)\right) \to 0 \tag{C.5}$$

From this sequence one can compute the Chern character of $\mathcal{O}_{D_1}(-D_1)$ and again up to the class of a point,

$$\mathrm{ch}\left[i_*\left(\mathcal{O}_{D_1}(-D_1)\right)\right] = \mathrm{ch}\left[R_2^1\right], \tag{C.6}$$

[7]Here we are treating both compact and non-compact divisors on par and we will do so throughout this section. A detailed technical justification of this is beyond the scope of this paper.

The R-sheaves for the Higgs branes of the $\mathbb{C}^3/\mathbb{Z}_5$ orbifold

One can carry out a similar exercise for the $\mathbb{C}^3/\mathbb{Z}_5$ orbifold. We will give below the sequences used to show that the Chern characters of the objects match with the Chern characters of the R-sheaves.

Type I. The sequences of interest are,

$$\begin{aligned} &0 \to \mathcal{O}_X(-D_1) \to \mathcal{O}_X \to i_*\left(\mathcal{O}_{D_1}\right) \to 0 \\ &0 \to \mathcal{O}_X(-2D_1) \to \mathcal{O}_X(-D_1) \to i_*\left(\mathcal{O}_{D_1}(-D_1)\right) \to 0 \\ &0 \to \mathcal{O}_X(-D_1-D_3) \to \mathcal{O}_X(-D_3) \to i_*\left(\mathcal{O}_{D_1}(-D_3)\right) \to 0 \end{aligned} \tag{C.7}$$

Using these sequences one can show that up to the class of a point,

$$\begin{aligned} \mathrm{ch}\left[i_*\left(\mathcal{O}_{D_1}\right)\right] &= \mathrm{ch}\left[R_2^{(0)}\right] , \\ \mathrm{ch}\left[i_*\left(\mathcal{O}_{D_1}(-D_1)\right)\right] &= \mathrm{ch}\left[R_2^{(1)}\right] , \\ \mathrm{ch}\left[i_*\left(\mathcal{O}_{D_1}(-D_3)\right)\right] &= \mathrm{ch}\left[R_2^{(3)}\right] . \end{aligned} \tag{C.8}$$

Type II. The sequences of interest are,

$$\begin{aligned} &0 \to \mathcal{O}_X(-D_3) \to \mathcal{O}_X \to j_*\left(\mathcal{O}_{D_3}\right) \to 0 \\ &0 \to \mathcal{O}_X(-D_1-D_3) \to \mathcal{O}_X(-D_1) \to j_*\left(\mathcal{O}_{D_3}(-D_1\right) \to 0 \end{aligned} \tag{C.9}$$

From the above sequences, one gets,

$$\begin{aligned} \mathrm{ch}\left[j_*\left(\mathcal{O}_{D_3}\right)\right] &= \mathrm{ch}\left[R_2^{(0)}\right] , \\ \mathrm{ch}\left[j_*\left(\mathcal{O}_{D_3}(-D_1\right)\right] &= \mathrm{ch}\left[R_2^{(1)}\right] . \end{aligned}$$

D. The Coulomb branes

In this appendix we will write down some representative objects for the Coulomb branes. These can be derived in the same way as the duals for the Higgs branes using appropriate sequences, which we don't write down. We write them down as sum of terms each corresponding to objects with support on both the non-compact divisor and on some compact divisor. We emphasise that these are not unique. The key point is that unlike the duals for the Higgs branes, these cannot be written as purely line bundles supported on the non-compact divisor.

Coulomb brane for the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold

$$R^3_{(2)} = i_*\left[\mathcal{O}_{D_1}(D_1)\right] + j_*\left[\mathcal{O}_{D_4}(-2D_1)\right] . \tag{D.1}$$

Coulomb branes for the $\mathbb{C}^3/\mathbb{Z}_5$ orbifold

Type I.

$$R^2_{(2)} = i_*\left[\mathcal{O}_{D_1}(-2D_1)\right] + j_*\left[\mathcal{O}_{D_4}(D_4)\right] , \tag{D.2}$$

$$R^4_{(2)} = i_*\left[\mathcal{O}_{D_1}(D_1)\right] + j_*\left[\mathcal{O}_{D_4}(-D_1)\right] + k_*\left[\mathcal{O}_{D_5}(D_1+D_5)\right] . \tag{D.3}$$

Type II.

$$
\begin{aligned}
R^2_{(2)} &= l_* \left[\mathcal{O}_{D_3}(D_3)\right] + j_* \left[\mathcal{O}_{D_4}(D_4)\right] + k_* \left[\mathcal{O}_{D_5}(-2D_1)\right] \ , && \text{(D.4)}\\
R^3_{(2)} &= l_* \left[\mathcal{O}_{D_3}(-D_3)\right] + k_* \left[\mathcal{O}_{D_5}(D_5 - D_1)\right] \ , \\
R^4_{(2)} &= l_* \left[\mathcal{O}_{D_3}(-D_3 - D_1)\right] + k_* \left[\mathcal{O}_{D_5}(D_5 - 2D_1)\right] \ . && \text{(D.5)}
\end{aligned}
$$

Here i_*, j_*, k_* and l_* are the push-forwards for the appropriate divisors.

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